

Chapter 4: Standing waves

In this chapter, I will discuss *standing waves*, which are stationary wave patterns that occur in nature all the time. They are particularly relevant in music, since both instruments using strings and instruments using air columns exhibit standing wave patterns. I will review a few key wave concepts from previous chapters, and then delve into how some of those wave properties produce standing waves.

Review

So far, we have seen the following:

- There are two kinds of wave disturbances we can classify: *transverse waves* where the medium travels perpendicular to the direction the wave disturbance travels, and *longitudinal waves* where the medium moves parallel (or opposite) to the direction the wave disturbance travels. Note: standing waves can be either type, as we'll see.
- A wave disturbance can travel a long distance compared to how far an individual piece of the medium moves.
- If v is the speed a wave travels, f is the wave's frequency, and λ is the wavelength, the quantities are related by the equation $v = f\lambda$.
- Waves can (in some cases) *reflect* off surfaces, *refract* (bend) in different media, and *interfere* with other waves.
- Interference is governed by the *superposition principle*: when two waves from different sources interact, the pattern that results is the algebraic sum of the two individual wave patterns.
- There are different ways waves can interfere. At the extremes are *constructive interference*, where two waves of equal wavelength line up and add to each other and *destructive interference*, where peaks of one wave line up with troughs of another and cancellation happens.

Standing wave patterns require understanding the last three of these bullet points especially.

Standing waves

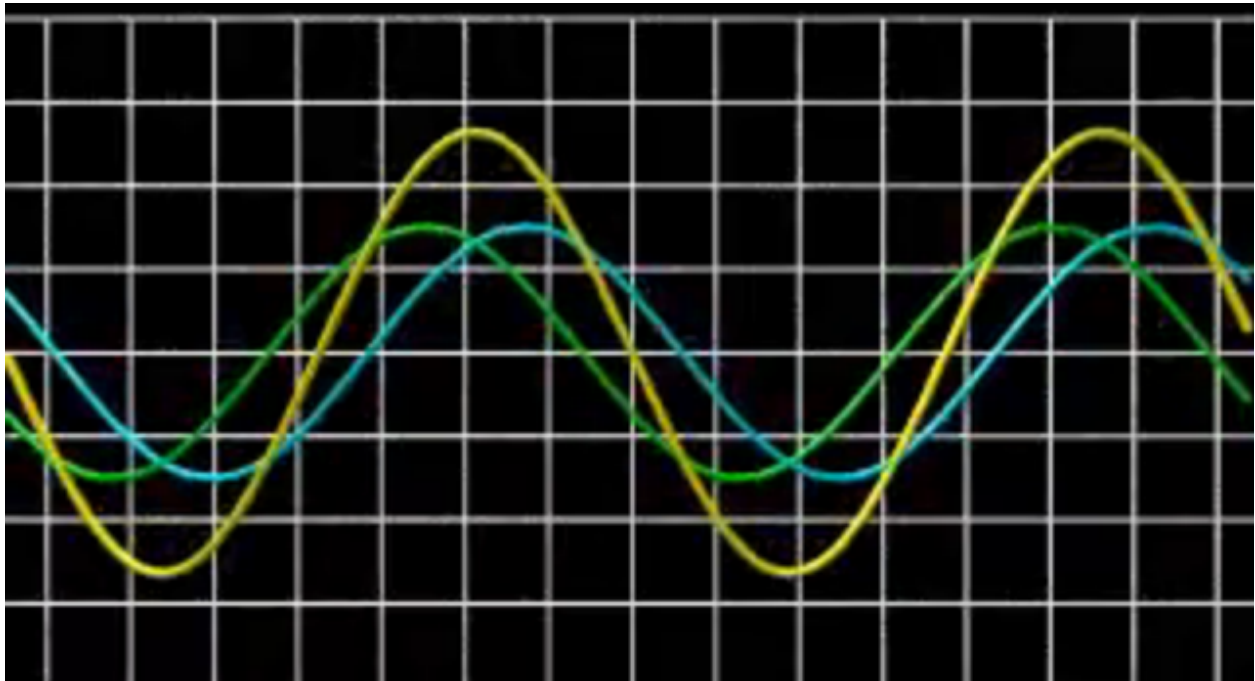
The best way to visualize a standing wave is to think of a stretched slinky or violin/guitar/ukulele/etc. string. If you shake or pluck one end, you'll get a wave pulse traveling down the length of the string. At the string end, a reflection will occur, and a pulse will return to you.

If you keep shaking or plucking one end, waves will leave your hand and head down the string *constantly*, and you'll also *constantly* be receiving reflections from the other end. One way to think

about it is that each end acts as a source of wave pulses. Therefore, the pulses can interact and *interfere* with one another.

What happens with this interference? A movie best illustrates this. If you send pulses with one frequency down a string, you'll get reflections of the same frequency. The following animation demonstrates this. View it now if you are able!

([Video link of a standing wave](#))



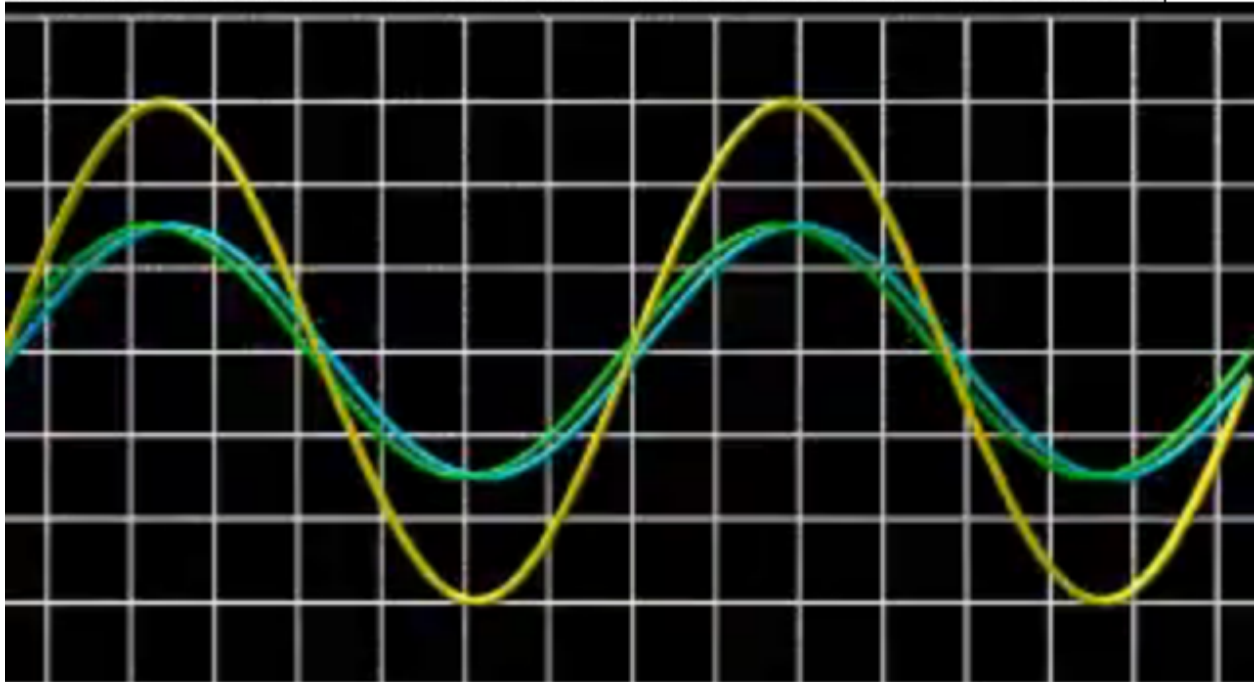
A screenshot of the standing wave movie

Here's what you are seeing: the green and blue waves represent pulses going one direction and the reflections going the other direction (which color is which does not matter). The yellow wave is the *superposition* of the green and yellow waves.

Compare the wavelengths of the three waves from the picture. What do you see?

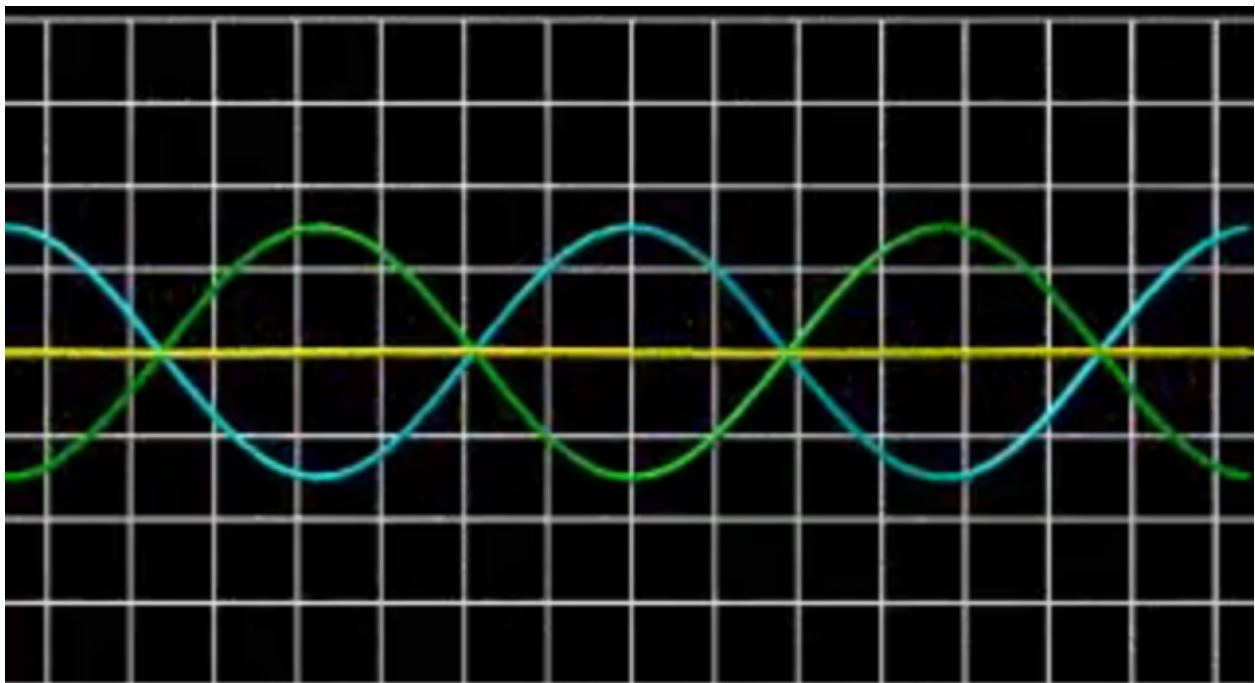
In reality, you never see the green and blue waves. All you end up seeing is the yellow result, so that's why it's important to look at the video – a real string or slinky would vibrate exactly as the yellow wave does.

Let's look at some particular moments in the animation:



Constructive interference between outgoing and reflected waves

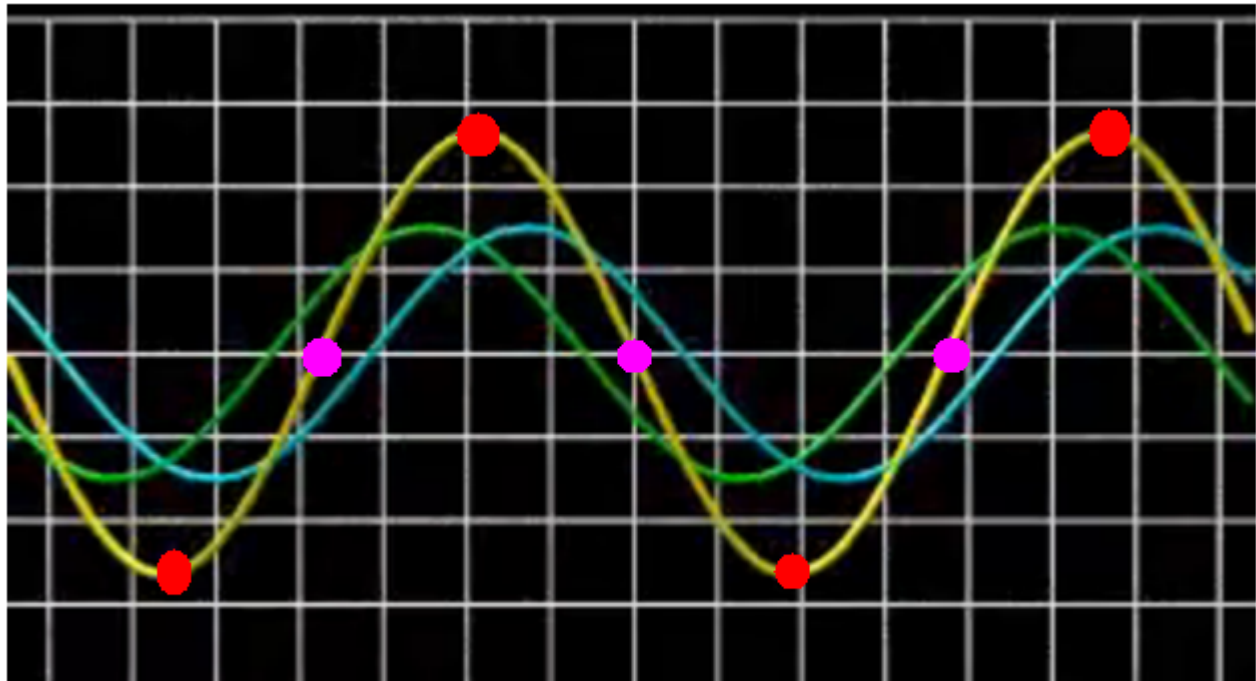
At this particular moment shown above, the green and blue waves line up, and the yellow wave is at a point of maximum amplitude. *Constructive interference* is happening everywhere along the string.



Destructive interference

At some prior (or later) time, the green and blue waves are now half a wavelength shifted with respect to each other, and *destructive interference* results. At this moment in time, the resultant wave is completely flat, and the string we're abstracting here is completely flat too. In motion, the string alternates in position between flat, maximum amplitude, and back again (covering all points in between).

There are special places along the standing wave pattern, and I've marked them below:



● nodes

● antinodes

Showing the locations of nodes and antinodes for a standing wave

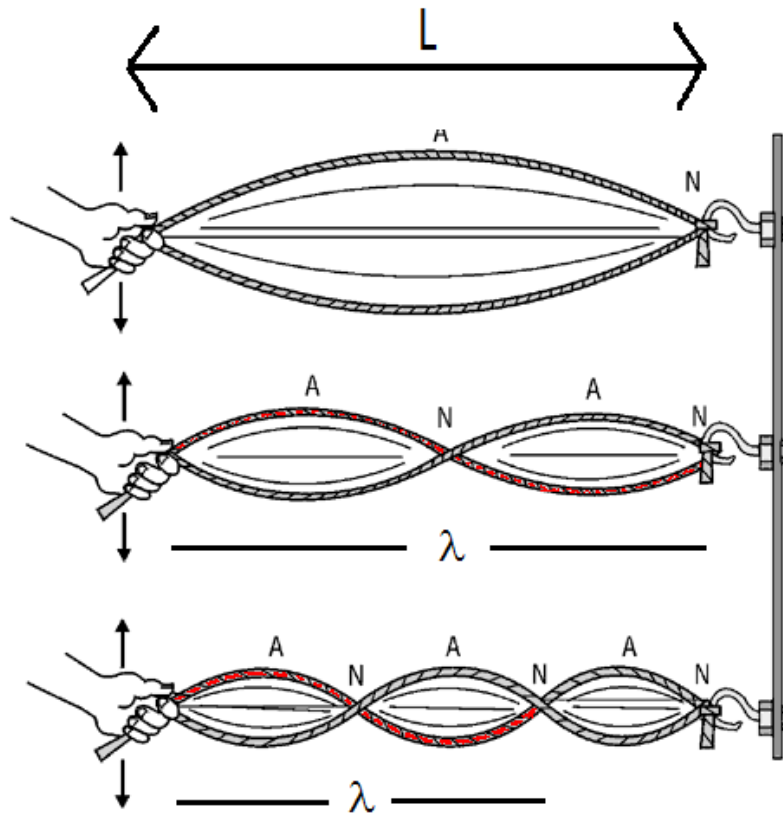
The parts marked in magenta/hot pink are called *nodes*. These are places that in this pattern do not move at all. In the animation, notice how those spots stay fixed. The places marked in red are called *antinodes*, and these are the places along the yellow wave that move up and down the most. Antinodes are locations where constructive interference is always happening, and nodes are places where destructive interference is always happening.

To see this, look the nodes. For the outer two cases, the green wave is “positive” and the blue is “negative.” In the center case, their roles are reversed. The main point is this: at these nodes, the green and blue waves oppose each other at *any time you observe the standing wave*. Similarly, the antinodes occur in places where the green and blue waves *never* cancel each other out, even a little.

In the next section, I will discuss how we can determine the wavelength of a standing wave.

Anatomy of a standing wave

If one sets up a standing wave on a string (especially one that moves quickly), it may look like one of the pictures below:



Three standing wave modes. In the lower two modes, one wavelength is highlighted in red.

Each of these patterns is called a *mode of vibration*. There are only certain allowed frequencies of vibrational modes for a string under a specific length and tension condition. One can get the higher modes by shaking the string faster, as shown in [this](#) animation.

If you have a strobe light, you can freeze a wave with a strobe light (INSERT VIDEO HERE) and see where the string is at any one moment. For two of the modes above, I have highlighted one full cycle of the wave. Notice that for the second case, one whole wave cycle – and it's a wavelength since we are looking at a picture of the wave in space – is equal to the length of the string. In the third case, we get through a whole wavelength by using only $2/3$ of the string.

In the top case, if you look at either the top or bottom image of rope, you will see you don't get to complete one full cycle, but half of one. For the top case, the wavelength of the standing wave is *double* the length of the string.

For all three cases, to get a wavelength, you have to take the distance across two loops of string. I might write this as:

$$\lambda = 2 \times (\text{distance across one loop}) \text{ or}$$

$$\lambda = 2(\text{NN})$$

where (NN) means the distance between consecutive nodes.

I'd like to be able to show you how you'd use this in a problem, but first let me give an example of where standing wave modes are taken advantage of in music.

String harmonics

If you look at the picture on the previous page, you see three ways in which a string can be excited in a standing wave pattern. Almost all guitar (and violin, cello, etc.) playing is done with the first mode primarily – the musician adjusts the note being played by either changing strings, fretting the instrument to create a shorter string, or adjusting the string tension (tuning). However, some musicians use *harmonics* to add colorful passages to a song. Even if you're not a musician, you've probably heard these before. A famous example from '70s-era rock is the [opening to Roundabout](#) (I posted a newer live version so you could watch Steve Howe play the harmonics). A newer instrumental song that uses a LOT more harmonics is [Reality](#) by Antoine Dufour. If you want to see it done on bass, you should watch [this clip](#) of Victor Wooten using them to play "Amazing Grace."

Most instrument fretting is done by holding the string down to the fretboard. Harmonics are made (as you can see in the videos) by lightly tapping the strings in special places. What's going on here is the musician is forming a node at certain frets and a high frequency standing wave forms. This is why harmonics are often higher pitched in comparison to other notes typically played on stringed instruments.

A schematic of some special harmonic locations is below.

If you lightly tap just around the 12th fret on a guitar, you form a node exactly halfway between the neck and the bridge. Then, when you strike the string, you force Figure 2 below to be the major vibrational mode. Figure 2 has half the wavelength as Figure 1, so it will have double the frequency, and the note you hear will be one octave higher than the unfretted string.

When we get to pitch and frequency later, I will discuss these harmonics in more detail and mention the *harmonic series* they form musically.

Fig.1

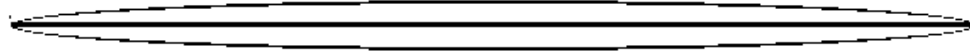


Fig.2



Fig.3

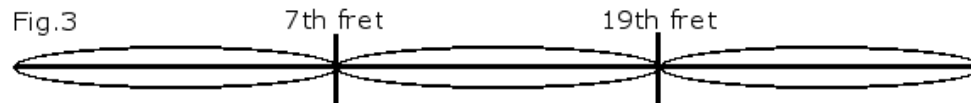
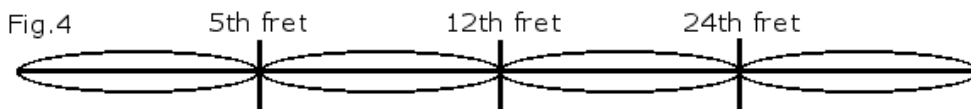


Fig.4



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Locations of harmonic nodes on a guitar string

The 7th and 5th frets on a guitar are good places to make harmonics as well because they produce notes triple and quadruple the frequency of the note played in Figure 1.

A mathematical example of standing waves

Suppose I was wondering how fast a wave in a guitar string travels. There's a way to figure that out!

The frequency of a guitar's A string is 110 Hz. If you fret the 5th fret like in Figure 4 above, you'll get a frequency of 440 Hz, which is the standard reference pitch for many orchestras.



My guitar string length is 65 cm, or 0.65 m. Now, if you want to think about it quickly, you can figure that since two loops makes one wavelength, I simply need to take half of this length (0.325 m) as my wavelength.

Alternatively, I can do this: there are four loops, so each loop is $65 \text{ cm} / 4 = 16.25 \text{ cm}$ long. Each wavelength is two loops ($\lambda = 2(NN)$), so the wavelength is $2 \times 16.25 \text{ cm}$ or 32.5 cm . The wave speed, then, is given by $v = f\lambda$ or $v = (440 \text{ Hz}) (0.325 \text{ m}) = 143 \text{ m/s}$. That's almost 320 mph!

Wave speed on a string

We can be even more quantitative about the wave speed on a string. It is given by the formula:

$$v = \sqrt{\frac{F_T}{\mu}}$$

where v is the wave speed, the **force of tension** on the string is F_T and the **mass density** is given by the Greek letter μ (mu). In SI units, F_T is given in Newtons (N), while mass density is trickier. You might know density as mass per unit volume from your previous studies in science, but this density is a mass per unit *length*, so its units in SI are kilograms per meter or kg/m.

This means that if wave speed is plotted on the y-axis of a graph and tension is on the x-axis, you shouldn't find a straight line. However, if you square both sides of this equation, you get $v^2 = F_T/\mu$, and that shows v^2 on the y-axis and F_T on the x-axis give a straight line with slope $1/\mu$. Mass density tends to be higher for, say, thicker strings. For example, the low E string on a guitar weighs a lot more than the high E string, even though they're the same length. This makes sense – lower strings are thicker.

Standing waves in pipes

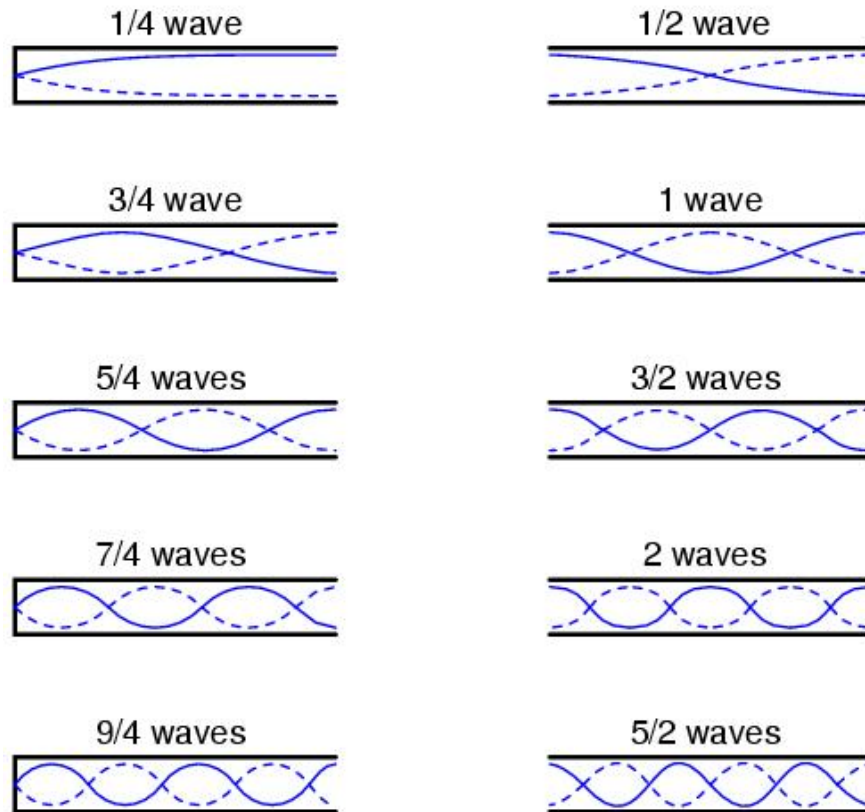
We cannot cover the full range of standing wave musical applications unless we discuss standing waves in pipes. The string examples in the previous section are good for (of course) the stringed instruments, but many brass and woodwind instruments employ slightly different standing waves.

Essentially, the key difference is in how one draws the waves. For a string, the two ends are fixed and therefore act as nodes. A pipe is different since for a good, resonant standing wave, we want the MAXIMUM possible air vibration at the open ends. Therefore, the open ends of a pipe act as antinodes!

Unfortunately, the pictures we often use to draw these waves become a bit more of an abstraction – since sound waves are longitudinal, and those are harder to depict in standing wave form, one must look at a transverse wave picture and picture longitudinal vibrations with similar node/antinode patterns.

Two sets of cases exist for pipe standing waves: pipes with one end closed and pipes with both ends open. I think it's best now that you see what these patterns look like (again, showing a transverse pattern for ease of illustration).

Standing sound waves in open-ended tubes



Pipe standing wave patterns ([internet source](#))

Observe the patterns in these pictures. The pipes with one closed end are on the left, and the open-ended pipes are on the right.

Activity / Discussion questions:

1. Look for patterns in the pictures on the left. One can fit different numbers of wavelengths in the pipes on the left. What pattern do you notice there?
2. Look for patterns in the pictures on the right. One can fit different numbers of wavelengths in the pipes on the right. What pattern do you notice there?
3. Do the wavelengths get shorter or longer as you move down the chart?
4. If the speed of sound in air is the same for all cases, what happens to the frequency of the waves as you move down the chart?

(answers on next page)

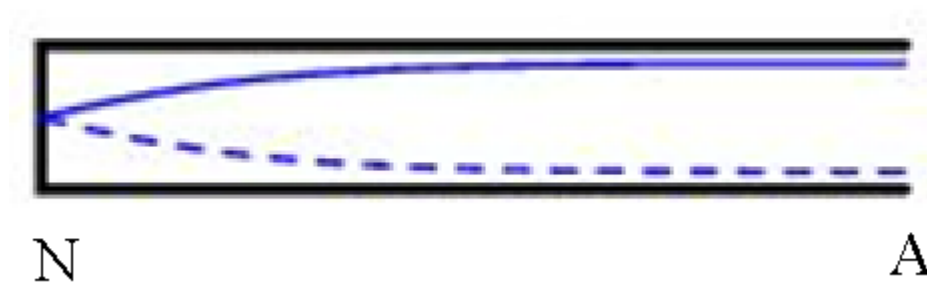
You can answer some of the above questions on your own right now. To get into the grittier mathematical details, let's work through some examples.

Activity / Discussion answers (from previous page):

1. Half-closed pipes always have an odd number of quarter-waves in them ($1/4$, $3/4$, $5/4$, etc.)
2. Open pipes have a whole number of loops in them (1, 2, 3, etc.)
3. Wavelengths get shorter since the distance between nodes and antinodes shrinks.
4. Frequencies go up as you go down the chart because for a given wave speed, wavelength and frequency are inversely related.

You may note that open pipes have a whole number of loops in them. The only difference is that one of the loops is comprised of a half-loop on each of the ends. It is therefore possible to find in a straightforward manner the size of one loop and then double it to get the wavelength.

The closed pipes are a little trickier. Observe the first one here:



I have indicated the node and antinode of this in the usual way. Suppose this pipe is 60 cm long and we are in an environment where the speed of sound is 343 m/s. We have half a wave loop here, so a full loop (if it were visible) would be 120 cm, and the wavelength of this sound wave would be double this, or 240 cm.

Since $f = v/\lambda$, the frequency of this sound wave would then be given by $(343 \text{ m/s}) / (2.40 \text{ m}) = \mathbf{143 \text{ Hz}}$.

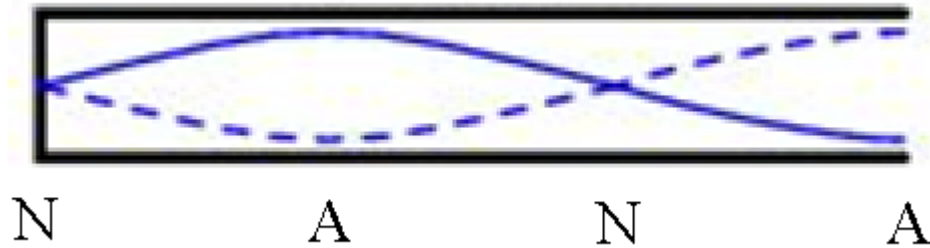
Note here that we consider this pipe to be $1/4$ of the full wavelength since two loops make up one wavelength and we have half a loop here (and one half is a quarter of 2). We can generalize this by saying:

$$\lambda = 4 \times (\text{distance across one half-loop}) \text{ or}$$

$$\lambda = 4(\text{NA})$$

where “NA” is the distance between a node and its adjacent antinode. “AN” and “NA” are conceptually equivalent if you are counting half loops.

Let’s do another example to show how this equation can be used in practice.



Assume the same conditions – a 60 cm pipe and sound traveling at 343 m/s. Note here that there are not one but THREE “NA” gaps going from left to right. You can also think of it as “three half loops” visible here. Therefore, each half loop is $60 \text{ cm} / 3 = 20 \text{ cm}$ long. “NA,” then, is 20 cm. From above,

$$\lambda = 4(20 \text{ cm}) = 80 \text{ cm or } 0.8 \text{ m.}$$

And again, since $f = v/\lambda$, the frequency of this sound wave would then be given by $(343 \text{ m/s}) / (0.80 \text{ m}) = \mathbf{429 \text{ Hz}}$. Note that the wavelength has dropped by a factor of three (240 cm to 80 cm) and the frequency has gone up by a factor of three (143 Hz to 429 Hz).

The closed pipe cases all follow this pattern – the first possible wave shown makes a *fundamental frequency*, and all subsequent cases give ODD multiples of that frequency.

Since the open pipes feature a whole number of loops in them, we get all the integer frequency multiples instead of just the odd ones. For example, in a 60 cm pipe, the first picture shown (and reproduced here)



has one loop of 60 cm length, therefore the wavelength is 120 cm. In room temperature air, the frequency would be 286 Hz (check this!)

The next pattern has two full loops, so the wavelength is 60 cm and the frequency is 572 Hz, which is double the fundamental frequency. You can get triple, quadruple, etc. the original frequency just by moving down the chart.

Why bother practically? Unlike a piano, where each key plays one and only one note, or a guitar where many notes can be played in multiple places, wind and brass instruments can play multiple notes for every fingering/valve position. The effect can be shown playfully in demos like [this one](#).

A resonance tube (basically, a cardboard pipe with one end closed) can be used to find the speed of sound in air if you know the frequency played into it. See [a Youtube video](#) demonstrating how this is done. This involves the same physics as the water resonance tube experiment written about [here](#).

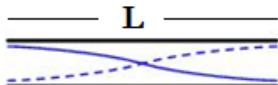
Formalizing the mathematics of pipe harmonics

In other texts on waves, it is not typical to talk about node-node separations as “NN” or node-antinode separations as “NA”. I find them useful ideas since visually it is relatively simple to count whole or half loops. However, a more formal approach is better for the purposes of writing compact formulas that can be used for different harmonics.

Suppose a string or pipe has a length L . The wavelengths that are allowed are below.

Strings and open pipes

I group these two together because mathematically these two behave the same – the wavelength of the fundamental standing wave is double the length of the string or pipe ($2L$), and the wavelength of the harmonics is $2L$ divided by the harmonic number, as shown in the figure:

	harmonic number n	wavelength
	1	$2L$
	2	$\frac{2L}{2} = L$
	3	$\frac{2L}{3}$
	4	$\frac{2L}{4} = \frac{L}{2}$
	5	$\frac{2L}{5}$

In general, we can say that $\lambda = \frac{2L}{n}$, where n is the *harmonic number* ($n = 1$ represents the fundamental).

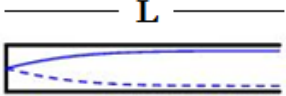
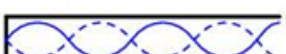
Also, since $v = f\lambda$ and $f = \frac{v}{\lambda}$, (again, v is the speed of the waves in whatever medium you have) we can say the possible harmonic frequencies of a string or pipe obey

$$f = v \times \frac{n}{2L} = \frac{nv}{2L}$$

The harmonic number n here can be 1, 2, 3, and so on. So $n = 1$ gives the fundamental frequency, then the harmonics are simply double, triple, quadruple, etc., that. We will see in a later chapter what notes these harmonics give.

Pipes closed on one end

I want to be consistent and use the symbol n for harmonic number – only this time, n has to be an odd number. An additional column in the figure below helps us see the pattern in allowed wavelengths. Notice that in the earlier figure with these pipes, the captions indicate whether each picture shows a quarter-wave, three quarters of a wave, and so on. I have just turned this into more formal math. The “L =” column shows how many wavelengths are pictured, then I solve for wavelength in terms of L .

	harmonic number n	L =	wavelength
	1	$\frac{\lambda}{4}$	$4L$
	3	$\frac{3\lambda}{4}$	$\frac{4L}{3}$
	5	$\frac{5\lambda}{4}$	$\frac{4L}{5}$
	7	$\frac{7\lambda}{4}$	$\frac{4L}{7}$
	9	$\frac{9\lambda}{4}$	$\frac{4L}{9}$

The wavelengths possible are $\lambda = \frac{4L}{n}$, where n is an **ODD** harmonic number. Similarly, we get

$$f = v \times \frac{n}{4L} = \frac{nv}{4L}$$

again, with n as an odd number. Sometimes instead of n you may see $(2m - 1)$ used where instead of m being odd, m takes the values 1, 2, 3, etc. – this is another way of writing an odd number.

A quick thing to notice: the harmonic frequency formulas are similar but the closed pipe frequency formula has a 4 in the denominator. Comparing the case of $n = 1$ for a closed and open pipe of the same length, we get that the frequency of the fundamental is twice as high for the open pipe. This is most easily seen on Boomwhackers (pictured), a toy that makes a note when a plastic tube is struck.



Boomwhackers

The tubes shown play a C major scale when “whacked” in order, but what the picture doesn’t show you is that they come with black plastic caps you can place over one end. If you put a cap on the end of the highest tube shown here, it sounds just like the lowest tube does with no cap.

Example: notes on a string

I have a small guitar (a Martin Backpacker) that’s portable for camping outings. It has only 15 frets or so and a slightly shorter-than-average 24” scale length (that’s about 60 cm). The high E string is supposed to play with a frequency of about 330 Hz and the F above that is about 350 Hz. Where would you expect the location of the first fret to be?

Here’s how to work it out. The frequencies I mentioned are for the fundamental mode, which means the wavelength of the waves is twice the length of the string. Therefore, the E note has a wavelength on this string of 120 cm or 1.2 m.

Why find that? We get the wave speed this way. Another equation exists for wave speed on strings, but I don’t happen to know the string tension or density without looking it up for this instrument. It’s simpler to use $v = f\lambda$ and get $v = (330 \text{ Hz})(1.2 \text{ m}) = 396 \text{ m/s}$.

The higher F note is played on the same string—same tension, density, everything. So we can find what wavelength makes the 350 Hz note: $350 \text{ Hz} \times \lambda = 396 \text{ m/s}$. We get about 1.131 m for wavelength, and the string length needed would be half this: 0.566 m or 3.4 cm less than the original length. That is almost exactly where the first fret on my guitar is!

Summary

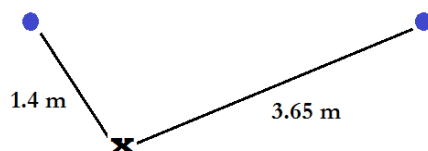
Standing waves are stationary wave patterns that occur when outgoing and reflected waves interact and interfere with each other. They are stationary in the sense that there are fixed *nodes* and *antinodes* that do not move. They can occur for both transverse and longitudinal waves.

To recap how to look at standing waves mathematically, remember the following:

- It helps to draw the situation you are given, identify the nodes and antinodes, and count the number of loops.
- One wavelength is equivalent to the distance across two standing wave “loops.”
- The wave speed equation still applies even though we realize standing waves come from interference of waves going in both directions..
- For harmonics on a string, all the modes vibrate at the same speed. Patterns with more visible loops have a shorter wavelength and thus a higher frequency.
- The wave speed on a string is a function involving the tension on the string and the *mass density* or mass per unit length of that string.
- In pipes, it is best to draw the situation and either find the size of a full loop or a half loop. Pipes closed at one end and open pipes have different harmonic patterns. Open pipes behave like strings in that all harmonics are possible. Pipes closed at one end only permit odd harmonics.

Homework problems

1. A string of length 0.9 m is oscillating with a certain frequency and sets up a standing wave with three loops. If the speed of the wave on the string is 45.0 m/s, do the following: (a) Draw this picture and label the nodes and antinodes, (b) find the frequency of the vibrator shaking the string (you'll need to find λ first), (c) find how long it takes one signal to travel the entire length of the string and back.
2. Find the length of one loop, the entire wavelength, and the speed of the wave v for the following three situations: (a) a string 120 cm long vibrating at 60 Hz with three loops (two nodes between the ends) (b) a string 80 cm long vibrating at 20 Hz with just an antinode in the middle (one loop only) (c) a 160 cm string vibrating at 30 Hz with four loops between the ends.
3. Consider a violin string of length $L = 31.6$ cm. Waves travel on this string with a velocity $v = 277$ m/s. (a) If you pluck it and there are no obstructions to the oscillation, what is the wavelength of this standing wave? (b) Is it possible to have a standing wave on this string with a larger wavelength? Explain your reasoning. (c) What is the largest period a standing wave can have on this string? (d) Use physical reasoning (no equations allowed) to determine the frequency of vibration of the sound wave in air.
4. (a) My guitar's D string vibrates with a frequency of 147 Hz if there are no nodes between the string endpoints. Estimate how long a wavelength of this vibration must be (you must first estimate the length of a guitar string), and find how fast the waves must be traveling along the string's length. (b) The higher "B" string plays a 196 Hz frequency (again – just one antinode in the center.) Find the wave speed on that string. Does it surprise you that it's faster than part (a)?
5. If two in-phase speakers in a room (drawn with dots here) play the same sound with wavelength 0.75 m, would the listener at the X hear nearly nothing, a loud sound, or something in between?



6. A violin has a scale length of 32.7 cm (we'll go with what Stradivari used). It is tuned to 440 Hz. If the violinist wants to play a note with a frequency of 494 Hz (this is a B – one whole step up), where should she place her finger to shorten the string correctly? (b)

Due to her finger size, if the finger placement can be off by about 0.5 cm in either direction, what is the percentage difference between the “sharp” and “flat” errors?

7. The pitch of a guitar note is related to the length L of the vibrating portion of the string and its tension. The length is determined by the finger position amongst the frets while tension is adjusted with the tuning pins. A player can also change the pitch by bending the string, which changes the tension by a lot while keeping the length basically the same. Suppose a note is bent from E to F, which requires a change of frequency of 5.9% (that is, $f_F = 1.059 f_E$). By what factor has the tension been increased?
8. Let's solve a real string problem using my own acoustic as an example. The scale length is $25 \frac{1}{2}$ " (this is the maximum vibrating length of the strings - let's call that 65 cm). The low E that's played has a frequency of 82.4 Hz. (a) Find the wavelength of this fundamental string vibration from the scale length. (b) What is the speed of waves on this guitar string? (c) The string specs say 24.95 lb of tension is appropriate. That's 111 N. Find the mass density of this string in kg/m. (d) **Only do if you're bored:** the D'Addario tension chart says that the string gauge I buy for this has a “unit weight” of .00056962 lb/in. Compare this with part (c) using $1 \text{ kg} = 2.2 \text{ lb}$ and $39.37 \text{ in} = 1 \text{ m}$.

Pipes

9. Fill in the blanks in the table below, showing your work (including a sketch for each). Assume the speed of sound in air is 343 m/s (the value for typical room temperature).

Type of pipe	Length of pipe (m)	Harmonic	Wavelength λ (m)	Frequency f (Hz)
Closed	0.45	Fifth		
Open		Second	1.2	
Closed	0.25		0.2	
Open		Third	0.4	
Closed		First	3.6	
Open	0.3		0.6	

10. Assume the speed of sound is 340 m/s. A long vertical tube is completely filled with water and it has a tap that can be used to drop the level of water inside. A tuning fork of frequency 880 Hz is held ringing over the open end of the tube. As water flows out, the air column inside grows longer. Calculate the first three air column lengths L_1 , L_2 , and L_3 for which resonance occurs (i.e., when the sound grows loud). Draw three

separate pictures that show the standing waves in each case, and label the locations of all the nodes and the antinodes.

11. A flute is designed to play its fundamental note at 264 Hz, when the air temperature is 15°C. Calculate the length of the flute and the frequencies of the second and third harmonics if the speed of sound in m/s is given by $331 + 0.6 T$ (where T is the Celsius temperature).
12. A pipe of length 150 cm is open at both ends and a standing wave is set up with only one node between the ends. If the speed of sound is 343 m/s, find the frequency of this standing wave.
13. Take the same pipe as in the previous problem (same speed of sound), and close one of the ends. Now find the fundamental frequency.